Influence Maximization over Strategic Diffusion in Social Networks

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***Abstract*—We study the problem of diffusion speed maximization over strategic diffusion, where individuals decide to adopt a new behavior or not based on a networked coordination game with their neighbors. For a variety of topological structures of social networks, we design polynomial-time algorithms that provide provable approximation guarantees. By analyzing three graph classes, i.e., Erdős-Rényi, planted partition and geometrically structured graphs, we obtain new topological insights, which does not exists in the literature for popular epidemic-based models. Our results first imply that for globally well-connected graphs, a careful seeding is not necessary. On the other hand, for locally well-connected graphs, their clustering characteristics should be intelligently exploited for good seeding, where seeding inside and intersection of clusters are important for such graphs having big and small clusters, respectively. We believe that these new insights will provide useful tools to understand and control the sociological evolution of innovations spread over large-scale social networks.**

I. Introduction

Recently, with the rapid growth of social network services, social networks become one of major routes for spreading new ideas, behaviors, and innovations across the world through person-to-person local interactions. In this trend, various research communities including computer science, economics, and sociology, have actively studied the diffusion of new information in social networks.

Diffusion models are broadly classified into epidemic-based or game-based ones. The epidemic-based diffusion models inherit an underlying assumption that people adopt a new innovation by just contacting others who already selected the innovation, e.g., independent cascade (IC) and linear threshold models (LT) in [2]. In such models, the authors in [2] first addressed *influence maximization* which selects limited initial adopters as *seeds* so that they maximize influence spread, i.e., number of adopters at the end of diffusion process, which inspired several following works, e.g., [3]–[5].

Different from epidemic style diffusion, people's behavior is often strategic when they decide to adopt or not the innovation, i.e., an individual follows the innovation only if it provides sufficient utility, which increases with the number of neighbors adopting the same choice (i.e., coordination effect) [6]–[9]. The game-based diffusion models capture such strategic behaviors and situations where people have incentive to select the same choices with their friends', and tend to hesitate to have different choices. This may occur in cases of selecting an operating system, e.g., Windows versus Unix, a cellphone, e.g., Android versus iOS, or a political stance, e.g., Republican versus Democratic.

In game-based diffusion models with bounded rationality of individuals (i.e., noisy best response dynamic), it has been proved that the number of innovation adopters increases and the entire social network eventually shifts to the innovation [10], [11]. Thus, maximizing influence spread as for epidemic-based models is meaningless since the innovation is finally widespread at the equilibrium. In this paper, we study how to choose seeds with a given budget to accelerate the speed of diffusion under game-based diffusion models, which we call *diffusion speed maximization*.

A. Challenges and Contributions

The diffusion speed maximization in game-based diffusion models has arguably more challenges than the influence maximizastion in epidemic-based ones. First of all, the diffusion speed, i.e., the convergence time to the equilibrium, is neither albegraic nor combinatorial formula. In [9], the authors studied the convergence time and they converted it into a combinatorial optimization problem using the theory of meta-stability of Markov chains [12]. Unfortunately, solving the optimization problem is still computationally intractable. Similar challenges are also discussed in the influence maximization with epidemic diffusion [2], [3]. However, in most cases of the influence maximization, a greedy algorithm guarantees constant approximation because of the submodular structure in the optimization. Whereas, the diffusion speed maximization is still intractable even if we ignore the difficulty of estimating the diffusion time because the diffusion time has none of submodular or supermodular structures.

In spite of the above hardness, we propose polynomial-time approximation algorithms for three graph classes, Erdős-Rényi, planted partition and geometrically structured graphs, which correspond to *(a)* globally well-connected, *(b)* locally well-connected with big clusters and *(c)* locally well-connected with small clusters, respectively. Also, we prove performance guarantees in terms of approximation ratio as well as complexity. Our contribution lies in providing new insights on how to seed individuals depending on the connection structure of underlying graph topologies in strategic diffusion models. We summarize the main insights in what follows:

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First, for globally well-connected graphs like Erdős-Rényi graphs, careful seeding is not crucially required because of the high symmetry and dense connectivity of the graphs. However, for locally well-connected graphs, it is necessary to intelligently exploit their clustering characteristics, where the network-wide diffusion time is governed by both intra-cluster diffusion and inter-cluster correlation. In sharp contrast to epidemic-based models, it turns out that in *(b)* intra-cluster diffusion becomes the dominant factor, as opposed to in *(c)* where inter-cluster correlation dominantly determines the network-wide diffusion speed. Thus, as described in Sections III-B and III-C, for planted partition graphs, we focus only on how to distribute the seed budget to each (big) cluster, while for geometrically structured graphs, the seeds are mainly selected from the border individuals to remove inter-cluster correlation.

B. Related Work

In [2], [3], the influence maximization was first studied as a combinatorial problem in linear threshold and independent cascade models where spread of innovation occurs much like epidemic. The authors discussed two challenges in the formulation: #P-completeness in estimating influence for a given seed set, and NP-completeness in selecting an optimal seed set. Remarkably, a greedy algorithm achieves at least $(1 - 1/e - \varepsilon)$ of the optimal influence where ε represents the inaccuracy in estimation of influence. The guarantee was proved using the maximization technique on the submodular set function in [13]. However, since the naive estimation of influence, e.g., Markov Chain Monte-Carlo with ε accuracy, or calculation of exact estimation, does not tend to scale with the network size, various heuristic-based scalable estimations have been proposed [14], [15]. In [14], [15], the methods utilizing tree and clustering structures of a graph were proposed, but with experimental performance evaluations.

As variations of the influence maximization in [2], [3], there were considerable works which captured the time-sensitive nature of the diffusion process [4], [5], [16]. In particular, [4], Chen et al. first addressed the time-sensitiveness and proposed modified LT and IC models by adding the contact process, in which the newly activated individual delays its infection chance until it meets its neighbors. Using the modified models, the authors formulated an influence maximization with time deadline and proposed a greedy algorithm motivated by [2], [3]. In [5], Goyal et al. generalized the influence maximization problem in LT and IC models as an optimization problem with three dimensions: influence spread, seed budget, and time deadline. Remarkably, in [16], Du et al. addressed not only the time feature but also scalability with performance guarantee, whereas the previous works either addressed the time feature, e.g., [4], [5], or proposed empirically scalable solutions [14], [15].

Regarding the influence maximization in game-based diffusion models, in [6], [17], [18], the authors considered only best-response dynamics and studied the conditions (of network topology and the payoff difference between old and the new technologies) on the existence of a small seed set, referred as the so-called "contagion set," under which all individuals adopt new technology. In [19], a noisy best response was considered with objective of maximizing the influence spread by choosing a seed set assuming that there exists a set of "negative individuals," and a greedy algorithm was proposed with only experimental evaluations. However, these works only considered the maximum number of adoptions at the end of the influence process.

II. Model and Formulation

A. Networked Coordination Game

The networked coordination game is played among a set V of n users on an undirected graph $G = (V, E)$ where E is the set of edges corresponding to social relationships among users. On each edge, two users at the ends play a pairwise coordination game with the payoff matrix given in Table I where each user i's strategy, x_i, can be one of new or old innovations, $+1$ and -1.

Payoff. We let $\boldsymbol{x} = (x_j \in \{-1, +1\} : j \in V)$, and $\boldsymbol{x}_{\text{-}i} = (x_j : j \in V \setminus \{i\})$ be the states (i.e., a strategy vector chosen by the entire nodes) of all and those except for i, respectively. Then, user i's payoff is summation of payoffs against i's neighbors $N(i)$, i.e., $P_i(x_i, \boldsymbol{x}_{\text{-}i}) = \sum_{j \in N(i)} P(x_i, x_j)$.

TABLE I. Two-person coordination game

P	$+1$	-1
$+1$	(a, a)	(c, d)
-1	(d, c)	(b, b)

On the payoff, we assumes that (a) there always exists coordination gain, i.e., $a > d$ and $b > c$, and (b) the innovation provides better coordination gain, i.e., $a - d > b - c$.

B. Diffusion Dynamics

We introduce the notion of *seed set* $C \subset V$, of which individuals are initialized by $+1$ and keep their strategy $+1$ forever. We assume that each non-seeded user updates its strategy at the arrival of its own independent Poisson clock with rate 1. In the update, user i's best strategy to maximize its payoff is +1 if $(a - d)|N^+(i)| \geq (b - c)|N^-(i)|$ and it is -1 if not, where we let $N^+(i)$ and $N^-(i)$ denote the sets of node i's neighbors adopting $+1$ and -1, respectively. To simplify, we can write the rational user i's response, i.e., its best response, as $\text{sign}\Big(h_i + \sum_{j \in N(i)} x_j\Big)$ where $h_i = h|N(i)|$ and $h = \frac{a-d-b+c}{a-d+b-c}$.

Noisy best response: Logit dynamics. In practice, people do not always make the rational decision. We model such behavior by introducing a small mutation probability that non-optimal strategy is chosen, often called noisy best response. A version of the noisy best response we focus on in this paper is *logit dynamics* [20]–[23] that individuals adopt a strategy according to a distribution of the logit form in which for the given state $\boldsymbol{x}$, user i selects the strategy $y_i \in \{-1, +1\}$ with the following probability:

$$\mathbb{P}_\beta(y_i|\boldsymbol{x}) = \frac{\exp(\beta y_i K_i(\boldsymbol{x}))}{\exp(\beta K_i(\boldsymbol{x})) + \exp(-\beta K_i(\boldsymbol{x}))}.$$

where $K_i(\boldsymbol{x}) = \frac{1}{2}\Big(h_i + \sum_{j \in N(i)} x_j\Big)$ that is proportional to payoff gain to choose +1 instead of -1 so that the parameter $\beta \geq 0$ quantifies noise in the dynamics. For example, $\beta = \infty$ corresponds to the best response.

Convergence to equilibrium. The above dynamics is also called the Glauber dynamics in the (truncated) Ising model, where the truncation occurs since seeds' states are fixed. For a given seed set $C \subset V$, $\boldsymbol{x}(t)$ at time t is a continuous Markov chain with the state space $\mathcal{S}_C = \{\boldsymbol{z} \in \{-1,+1\}^V \mid z_i = +1 \text{ if } i \in C\}$. Also, this Markov chain is time-reversible with the stationary distribution $\mu_\beta(\boldsymbol{x}) \propto \exp(-\beta H(\boldsymbol{x}))$ where

$$H(\boldsymbol{x}) = -\frac{1}{2}\Bigg\{\sum_{(i,j)\in E} x_i x_j + \sum_{i \in V} h_i x_i\Bigg\} + (1+2h)|E|. \quad (1)$$

In the above, the constant term $(1+2h)|E|$ is just added for notational convenience. We note that $-H$ is often referred to as a *potential* function of the networked coordination game and H is called the *energy* function in literature.

Since the energy function minimizes at the state $\mathbf{+1}$ where all users' states are $+1$, as the users become more rational, i.e., $\beta \to \infty$, the stationary distribution concentrates to $\mathbf{+1}$. Thus, $+1$ becomes widespread after enough time to converge the equilibrium. The convergence was observed in [10], [11] under a similar noisy best response dynamics. Topological impact on the convergence time was studied by Montanari and Saberi in [9].

C. Problem Formulation

Our aim is to minimize the convergence time by seeding within a given seed budget. First, we define the *diffusion time* for a given seed set C:

$$\tau_+(C) = \sup_{\boldsymbol{y} \in \mathcal{S}_C} \inf\left\{t \geq 0 \mid \mathbb{P}_\beta\{T_+(C,\boldsymbol{y}) \geq t\} \leq e^{-1}\right\}$$

where $T_+(C,\boldsymbol{y})$ is the convergence time or the hitting time to $\mathbf{+1}$ starting from $\boldsymbol{y}$. The diffusion time provides upper bound of the time to widespread of innovation with probability greater than $1/2$.

Limit behavior of the diffusion time. As mentioned earlier, estimating the diffusion time is a highly non-trivial task since it is neither algebraic nor combinatorial. To overcome such challenge, we resort to meta-stability analysis by focusing on the small noise, i.e., $\beta \to \infty$, as in the other sociology literature, e.g., [9], [11]. As the similar result with [9], [12], we have the following limit behavior of the diffusion time that is aware of truncation by a given seed set $C \subset V$:

$$\tau_+(C) = \exp(\beta \Gamma^*(C) + o(\beta)), \qquad \text{as} \quad \beta \to \infty., \quad (2)$$

where we refer to $\Gamma^*(C)$ as the *diffusion exponent* with respect to the seed set C. In the above, $\Gamma^*(C)$ is defined as

$$\Gamma^*(C) = \max_{w_0 \in \mathcal{S}_C} \min_{\underline{w}: w_0 \to \mathbf{+1}} \max_{t < |\underline{w}|} [H(w_t) - H(w_0)] \quad (3)$$

where the minimization is taken over every possible path $\underline{w} = (w_0, w_1, \cdots, w_T = \mathbf{+1})$ such that for each t, w_t and w_{t+1} are same except for one coordinate.

Γ^* can be interpreted as the smallest "energy barrier" among all possible paths from the worst initial state to $\mathbf{+1}$. Also, it dominates the exponent of diffusion time as $\beta \to \infty$. In addition, it is known [9] that the minimization of (3) is achieved just at a *monotone* path on which a user is not allowed to take back from $+1$ to -1. Thus, such paths with the smallest barrier represent major diffusion patterns.

Problem formulation in combinatorial optimization. The formula (2) expresses $\tau_+(C)$ in terms of $\Gamma^*(C)$. We are motivated by this and we will focus on the following optimization to minimize the diffusion time with given seed budget k

$$\min_{C \subset V} \Gamma^*(C) \quad \text{subject to } |C| \leq k, \quad (4)$$

where, by (2), it becomes identical to a direct formulation, $\min_{C \subset V} \tau_+(C)$ subject to $C \leq k$.

We note that calculating $\Gamma^*(C)$ is still computationally infeasible even though it seems easier than computing $\tau_+(C)$. In addition, except such difficulty, for our formulation, we cannot apply the well-known optimization technique for submodular and submodular objectives in [13], [24] because our objective $\Gamma^*(\cdot)$ is neither submodular nor supermodular, whereas such techniques were used for influence maximization in epidemic-based models, e.g., [2]–[5]. These challenges motivate our study of a different kind of approximation scheme.

III. Main Result

In this section, we describe our polynomial-time approximation algorithms for the seeding problem (4). Each algorithm provides the guideline on which nodes should be seeded for fast diffusion over a game-based diffusion model for each of three graph classes having different topological structures in terms of connectivity and the degree of clustering. To this end, we first introduce the following notion of "approximate solution".

Definition 3.1: A seed set $C \subset V$ with $|C| \leq k$ is called a (γ, δ)-approximate solution of the seeding problem (4) if

$$\Gamma^*(C) \ \leq \ \gamma \cdot \min_{C': |C'| \leq \delta k} \Gamma^*(C'),$$

where $\gamma \geq 1$ and $\delta \leq 1$.

For a given seed set, we measure its degrees of suboptimality in *objective value* and *budget* in the parameters γ and δ, respectively. One can observe that the solution with $(\gamma, \delta) = (1, 1)$ corresponds to an optimal solution. In the rest of this section, we characterize approximate solutions for three graph classes which are distinguished by topological aspects on the criterion on how globally and locally well-connected nodes are.

A. Erdős-Rényi Graphs

We first consider *Erdős-Rényi (ER) graph,* denoted by $G_{\text{ER}}(n,p)$, consisting of n nodes of which each node pair has an edge with probability p. Let $\lambda = np$, roughly corresponding to the average number of neighbors per node, where our focus is when $\lambda = \omega(1)$. For ER graphs, we obtain the following result.

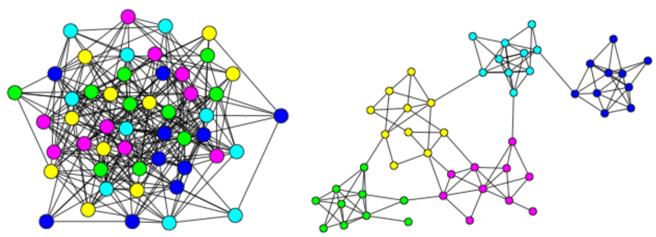

Fig. 1. An instance of ER-graph (left) and planted partition graph (right). Source: Lecture note of the network analysis and modeling course in Santa Fe Institute [25].

Theorem 3.1: For a ER graph $G_{\text{ER}}(n, p)$ and the seed budget k, every $C \subset V$ with $|C| = k$ is almost surely a (γ, δ)-approximate solution as $n \to \infty$, where $\lambda = \omega(1)$,

$$\delta = 1 \quad \text{and} \quad \gamma = 1 + \varepsilon \quad \text{for any given } \varepsilon > 0.$$

Theorem 3.1 implies that an *arbitrary* seed set C is, somewhat surprisingly, an almost optimal solution, i.e., $(\gamma, \delta) \to (1, 1)$. In addition, in the convergence, we can take $\varepsilon = O(\frac{1}{\sqrt{\lambda}})$. This implies that more careful seeding mechanism is requied for less (globally) well-connected graph. The proof of theroem relies on the observation that the energy function depends on only number of $+1$ becuase each node can have an edge to any node with equal probability, i.e., the graph structure is highly symmetric.

B. Planted Partition Graphs

As a generalized version of ER graphs, we consider the *planted partition graph* [1] It is a popular model, e.g., [26], for social networks with underlying clustered structure where several sets of nodes organize into densely linked communities.

We let $G_{\text{PP}}(n, p, q, \boldsymbol{\omega})$ denote the planted partition graph with a disjoint partition of the clusters $\{V_1, ..., V_m\}$, with $\bigcup_{l=1}^{m} V_l = V$, where we let $\boldsymbol{\omega} = (\omega_1, ..., \omega_m) \in (0, 1)^m$ denote the fraction of nodes in the graph that belongs to a cluster l be $\omega_l = |V_l|/n$. For a pair of $i, j \in V$, an edge (i, j) exists between them with probability $p = \Theta(1)$ for the nodes i and j if i, j belong to a same cluster, and with probability $q < p$, otherwise. We obtain the following result.

Theorem 3.2: For a planted partition graph $G_{\text{PP}}(n, p, q, \boldsymbol{\omega})$ and the seed budget k, every $C \subset V$ such that

$$C \in \arg \min_{\{C' : |C'| \le k\}} \max_{1 \le l \le m} \left(\frac{1 - h}{2} |V_l| - |C' \cap V_l| \right) \quad (5)$$

is almost surely a (γ, δ)-approximate solution as $n \to \infty$, where $q/p = o(1)$,

$$\delta = 1 \quad \text{and} \quad \gamma = 1 + \varepsilon \quad \text{for any given } \varepsilon > 0.$$

Theorem 3.2 provides an insight on how to allocate seeds, coming from solving a "simple" min-max optimization (5) whose computational complexity is $O(1)$. Intuitively, if the resulting seed set C in (5) allocates seeds proportionally to the size of each cluster, such seed set C is an almost optimal solution, regardless of how to seed inside each cluster.

To provide a sketch of proof for Theorem 3.2, observe that the dependency among diffusion processes in clusters can be ignored due to a few inter-cluster edges comparing to intra-cluster edges, i.e., $q/p = o(1)$. Hence, the entire diffusion process can be viewed as parallel processes of m ER graphs and diffusion speed of the process is determined by the slowest process among the parallel processes. This requires balance in allocating seeds to each cluster as expressed in (5). We note that if we greedily select seed based on node degree, some clusters where nodes have low degrees in average may starve so that the seeding performs poor due to the starvation clusters.

C. Geometrically Structured Graphs

Third, we consider locally well-connected graphs with *small* clusters. Those graphs include geometrically structured graphs such as planar and d-dimensional graphs. In these graphs, the inter-cluster correlation dominantly determines the network-wide diffusion speed, and hence seeds should be selected with goal of removing the correlation. One of achieving such a goal is to seed the *border nodes* among small clusters. Motivated by this, we design a generic algorithm, called PaS (Partitioning and Seeding)[2] for finding good seeds. As the name implies, PaS has two phases: (i) partitioning and (ii) seeding, as elaborated in what follows.

(i) Partitioning phase: In this phase, PaS finds a partitioning with a finite number of node clusters, where the number of clusters are chosen appropriately, depending on the underlying graph topologies. Except for a special cluster, say V_0, which will be used as the initial seed set, PaS will find the seeds contained in each cluster by the seeding phase.

(ii) Seeding phase: In this phase, PaS runs in multiple rounds, where it starts from the initial seed set V_0 and the seed set C increases by one in each round, until the entire seed set size becomes the target budget k. Let G_l and C_l be the subgraph induced and the seed contained, by l-th cluster V_l, respectively. In each round, the algorithm finds the partition l^* that has the largest Γ^* among the subgraphs $\{G_l\}_{l=1,...,m}$ and then with one more budgets for the slowest cluster l^*, it exhaustively chooses a seed set in the cluster which minimizes Γ^*.

Now, we are ready to present the performance guarantees of PaS. To that end, we introduce a notation: E_l is the edge set of the subgraph induced by $V_l \cup V_0$, where V_l is the l-th cluster resulting from the partitioning phase.

Theorem 3.3: For given graph $G = (V, E)$ and seeding budget $k = \kappa n$ with $\kappa \in (0, 1)$, suppose that $\{V_l : l =$

[1]This is often referred to as a stochastic block model.

[2]For a formal description, see **Algorithm 1** in [1]

$0, 1, \ldots, m\}$ in the partitioning phase of the PaS algorithm has the following condition:

For some $\varepsilon \in (0, 1)$,

$$|V_0| \leq \varepsilon n \quad \text{and} \quad |V_l| = O(1), \quad \text{for all } l = 1, ..., m. \quad (6)$$

Then, PaS outputs a $(1, 1 - \frac{\varepsilon}{\kappa})$-approximation solution and its seeding phase takes $O(n^2)$ time.

The condition (6) does not always hold. However, for d-dimensional graph, e.g., random geometric graph, and planar graph, e.g., 3-map graph, polynomial-time algorithms are known for computing such a partition satisfying the condition for *any* $\varepsilon = \Omega(1)$ [27] [3].

IV. Conclusion and Future Work

In this paper, we have studied the question on how the diffusion speed of a new innovation can be maximized under a noisy game-based model, by seeding a subset of individuals (within a give budget), i.e., convincing them to pre-adopt a new innovation. By analyzing three graph classes, i.e., Erdős-Rényi, planted partition and geometrically structured graphs, we obtain new topological insights, which does not exists in the literature for popular epidemic-based models. Our results first imply that for globally well-connected graphs, a careful seeding is not necessary. On the other hand, for locally well-connected graphs, their clustering characteristics should be intelligently exploited for good seeding, where seeding inside and intersection of clusters are important for such graphs having big and small clusters, respectively. We believe that these new insights will provide useful tools to understand and control the sociological evolution of innovations spread over large-scale social networks.

We have so far focused on how to maximize diffusion speed where individuals strategically behave for given social networks. For future work, we propose to address two issues coming from our assumption and goal: (i) influence maximization in more practical social graphs, and (ii) different purpose of controlling influence spread. First, in reality, our social relationship is time-varying and it is sometimes hidden even. This uncertainty of social graph inspires us to study a different kind of influence maximization. Second, we often observe negative results from the spread of false rumor, misinformation, and computer virus. This motivates to study diffusion minimization which hampers diffusion rather than encourages it as the present work.

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[3] In particular, the author [27] considers polynomially-growing graphs and minor-excluded graphs, where d-dimensional graphs and planar graphs are their special cases, respectively.