On the Impact of Global Information on Diffusion of Innovations over Social Networks

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***Abstract*—This paper studies how global information affects the diffusion of innovations on a network. The diffusion of innovation is modeled by the logit dynamics of a weighted N-person coordination game among (bounded) rational users where innovations spread through users' strategic choices. We find a critical asymptotic threshold for the weight on global information where the diffusion of innovations undergoes a transition in the rate of convergence regardless of any network structure. In particular, it is found that the convergence to the pervasive adoption is slowed down by global information.**

I. Introduction

The proliferation of online social networks has drastically changed how people interact and share information over the Internet. People are actively using social networks to get new information, exchange new ideas or behaviors, and adopt new innovations. As the role of the social networks in exchange and sharing of information increase, understanding the diffusion of information, behaviors or opinions, and innovations in the social networks becomes an important research issue, where diffusion by local interaction is the most prominent feature.

There exist two models studying the diffusion dynamics via local interactions in the social networks: (i) epidemic model and (ii) game-theoretic model. In epidemic models, people obtain new information and adopt new technology when they have just a contact with others who already had the new technology, i.e., the information or the technology innovation spreads like contagious disease [1], [2]. In game-theoretic models, a user adopts a new technology only if the new technology (behavior) maximizes her utility, which increases with the number of neighbors who adopt the same technology [3], [4], [5]. In both models, the major interests lie in (a) what is an equilibrium (or limiting behavior) of the diffusion process after long interactions and (b) how long does it take to reach an equilibrium, and how does such convergence depend on the topological properties of the given social network? Under a game-theoretic model, we characterize the role of global information for the question (b): how the global informaiton affects the convergence time to reach an equilibrium on general networks.

It is true that people are more affected by those who are in close relationship, which is the reason why study on local interactions of social networks is important. However, for better decision making, people generally want more information than the local information obtained from those who are in close relationship i.e., they use global information as well as local information. Indeed, one's adoption of new technology (for example, an upgrade to a new OS release) relies on not just the information from people that she knows, but also the global information that can summarize the information as a whole, e.g., public information/rumor in the media like news papers or TV broadcasting.

Motivated by the above, this paper characterizes the role of global information under a game-theoretic model and addresses the question of how the global informaiton affects the convergence time to reach an equilibrium on general networks. To that end, we start with an N-person coordination game among bounded rational users who sometimes choose a non-optimal decision. Users' decision making is modeled by the logit dynamics. To model global information, we use a simple weight value $p < 1$ that measures the strength of the global information on users' decision making. Our main contributions are summarized as:

- *Supercritical regime.* For $p > \frac{m_0}{N}$, the convergence time to the state of pervasive adoption of a new technology takes exponential time, where N is the number of total users and m_0 is a constant depending on the payoff structure and user's rationality, if the quality difference between the entrant and the incumbent, denoted by h, is not significantly large. This implies that the global information slows down the spread of information regardless of a graphs structure whenever $p > O(1/N)$.
- *Subcritical regime.* For $p < \frac{m_1}{N}$ with some constant m_1, the convergence time to the state of pervasive adoption of a new technology with global information is bounded by that without global information having smaller h' than h.

Our results imply that there exists a *phase transition* effect that when p is in the larger order than $1/N$, the convergence time becomes long *irrespective of* the topology of the underlying social network, but when p is in the smaller order than $1/N$, the convergence time to the pervasive adoption resembles that without global information.

II. Related Work

Game-theoretic models start with N-person coordination game which is an extension of 2×2 coordination game whose payoff is given by Table I. Note that 2×2 coordination game is a famous example for multiple Nash equilibria; the Nash equilibria are $(+1, +1)$ and $(-1, -1)$. Then, it is easy to check that the N- person coordination game also has two equilibria, all choosing $+1$ and all choosing -1. A new information, behavior or technology is represented by $+1$ and the diffusion of innovation is the spread of $+1$ among users.

TABLE I
2×2 COORDINATION GAME: $a > d,\ b > c,\ a - d > b - c$

	$+1$	-1
$+1$	(a, a)	(c, d)
-1	(d, c)	(b, b)

The multiplicity of Nash equilibrium has raised a question of equilibrium selection, which is extensively studied in economics through studying the limiting behavior of N-person coordination game. Equilibrium selection is equivalent or identical to the question in epidemic modeling that under which condition the disease is endemic. In the seminal paper [6], Kandori et al. studied the limiting behavior of N-person coordination game among bounded rational users who choose a non-optimal strategy with small deviation probability. They showed that the N-person coordination game converges to the state that every player chooses the strategy $+1$ if $a > b$ (and $c = d = 0$) as the deviation probability goes to 0.

Despite the fact that the long-run equilibirum is all choosing $+1$ as in [6] for the case $a > b$ and $c = d = 0$, Ellison [4] found that the time to reach the equilibrium depends much on the underlying graph structures. He considered two graphs, a complete graph and a ring graph, for which he obtained asymptotic upper bounds of the convergence time to the equilibrium as N goes to ∞; on a ring graph, the convergence time to the state of all $+1$ is bounded by a constant independent of the number of users, while it takes exponential time on a complete graph. Montanari et al. [7] extended Ellison' results to general graphs and found that the convergence time to the equilibrium is long if a graph is well-connected. The result is quite striking since it contrasts with the result in an epidemic model that the time of endemic spread is small for well-connected graphs. [1], [7].

The impact of global information on the dynamics of an evolutionary game or learning has been recently studied in [8], [9]. They studied how to use global information to guide players to choose a socially optimal (preferable) equilibrium when there are multiple equilibria. The global information is managed and broadcast to users by a central authority. A user may make her choice with global information but she may not trust the announced information and as a result she may ignore it. In contrast to their works, we do not assume that the global information is managed and announced by a centralized authority, and focus on the convergence time to an equilibrium. The global information we consider in this paper is a statistics of popularity which is available by easily accessible media like newspapers or TV broadcastings. In reality, it is what people want to know other than local neighbors' information for better decision making. Hence, they "voluntarily" take into account the global information.

III. System Model and Problem Formulation

A. Network Model and Game

Consider an undirected connected graph represented by $G = \{V, E\}$ where V is the set of nodes and E is the set of edges. Each node represents a player (or a user) and each edge represents a social relationship between two nodes forming the edge. We assume that there are N players i.e., $|V| = N$ and two technologies $+1$ and -1, representing a new entrant technology and an incumbent one, respectively. The choice of user i is denoted by x_i and the set of neigboring nodes by $\mathcal{N}_i = \{j \in V \mid (i,j) \in E\}$. We assume that all users adopt the incumbent -1 at the initial time and the entrant $+1$ provides better quality or aspects for users than the incumbent -1. We consider the situation where players are rational and repeatedly choose a technology.

We assume that a user is able to access global information such as total number of users, N, and the number of users choosing $+1$, i.e., $N^+ = |\{j \mid x_j = +1 \text{ and } j \in V\}$. Such global information can be obtained from newspapers or TV broadcasting. Since the global information is given as a collective statistics, each user does not know the strategies chosen by her non-neighbors. As opposed to global information, a user observes what strategy her neighbors choose. Thus, a user knows the number of neighbors choosing +1, N_i^+, i.e., $N_i^+ = |\{j \mid x_j = +1 \text{ and } j \in \mathcal{N}_i\}|$. It is clear that $N^+ \geq N_i^+$.

The payoff of a node is modeled to be given by putting weight q_1 on local information (her neighbors' choice) and weight q_2 on the global information. That is, a node i's payoff is given by:

$$\begin{aligned} & u_i(x_i, x_{-i}) \\ &= q_1 \sum_{j:(i,j)\in E} g(x_i, x_j) + q_2 \Big(N^+ g(x_i, 1) + (N - N^+) g(x_i, -1)\Big) \\ &= q_1 \sum_{j:(i,j)\in E} g(x_i, x_j) + q_2 \sum_{j:j\neq i} g(x_i, x_j) \\ &= (q_1 + q_2) \sum_{j:(i,j)\in E} g(x_i, x_j) + q_2 \sum_{j:(i,j)\notin E} g(x_i, x_j) \end{aligned}$$

where $x_{-i} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_N)$ and $g(x_i, x_j)$ is a payoff function of the 2×2 coordination game in Table I; $g(+1, +1) = a, g(+1, -1) = c, g(-1, +1) = d, g(-1, -1) = b$. As discussed earlier, we assume that that $a > d$, $b > c$ and $a - d > b - c$.

By normalizing, we have for some $0 < p < 1$, [1]

$$u_i(x_i, x_{-i}) = \sum_{j\in\mathcal{N}_i} g(x_i, x_j) + p \sum_{j\notin\mathcal{N}_i} g(x_i, x_j). \quad (1)$$

Also note that $p = 0$ corresponds to the case of no global information, i.e., there are only purely local interactions as in [7]. We easily see that there are two Nash equilibria $+\underline{1} = (+1, +1, ..., +1)$ where all players choose $+1$ and $-\underline{1} = (-1, -1, ..., -1)$ where all players choose -1.

[1] Note that a user may be more influenced by her neighbors than the anonymous, in which case a node would put more weight on local information than the global information. Then, we have $p < 1/2$ (since $q_1 > q_2$ and $\frac{q_2}{q_1+q_2} < 1/2$). However, this paper allows the case when more weight can possibly be assigned to global information, i.e., $0 < p < 1$.

B. Diffusion Dynamics

We now turn to how a user dynamically change their state (i.e., the technology that the user chooses over time). We assume that each user has her own Poisson random clock with unit rate and updates her strategy whenever it ticks.

We first describe the best response dynamics and then describe a noisy response dynamics which is of our main consideration. In the best response dynamics, a player i chooses the *best-response* strategy at time when her clock ticks, i.e., she chooses $+1$, if

$$u_i(+1, x_{-i}) \quad \geq \quad u_i(-1, x_{-i}), \tag{2}$$

otherwise, she chooses -1. Note that (2) is equivalent to:

$$(1-p)(N_i^+ + N_i\eta) + p\Big(N^+ + (N-1)\eta\Big) \quad \geq \quad 0$$

where $\eta = \frac{c-b}{a-d-c+b}$ and N_i is the number of neighboring nodes of i, i.e., $N_i = |\mathcal{N}_i|$. Then, by letting

$$K_i(x) \quad := \quad \sum_{j \in \mathcal{N}_i} x_j + p \sum_{j \notin \mathcal{N}_i} x_j + (1-p)h_i + ph_0,$$

where

$$h \quad = \quad \frac{a-d-b+c}{a-d+b-c}, \quad h_0 = h(N-1), \quad h_i = hN_i,$$

we see that $+1$ is the best response for i if and only if $K_i(x) \geq 0$, since (2) is equivalent to $K_i(x) \geq 0$.

In this paper, we consider a noisy version of the best response, called the noisy response dynamics. This is motivated by the facts that a user does not always make the best decision and in many cases, a user sometimes choose a non-optimal strategy with small probability, for the following reasons: (i) some users do not react instantaneously to their environment, (ii) users may have misinformed, and (iii) there is a small probability that agents behaves at random (mutation or experimentation).

A special case of noisy response dynamics is *the logit dynamics,* where each player i probabilistically chooses strategy $y_i \in \{+1, -1\}$ under given $\underline{x}$, with probability given by

$$\Pr(y_i|\underline{x}) \quad = \quad \frac{\exp(\beta y_i K_i(\underline{x}))}{\exp(\beta y_i K_i(\underline{x})) + \exp(-\beta y_i K_i(\underline{x}))}. \tag{3}$$

with $\beta > 0$. Note that if $\beta = 0$, a user chooses her strategy uniformly at random and if $\beta = \infty$, a user chooses the optimal strategy (i.e., best response).

With the probability of strategy selection as governed by (3), the entire system can be viewed as a continuous-time Markov chain on the state space $S = \{\underline{x} = (x_1, ..x_i, ..., x_N) \mid x_i = -1 \text{ or } 1\}$, where a state is a strategy profile $\underline{x}$. The transition probability from the state $\underline{x}$ to $\underline{y}$ with $y_j = x_j$ for $j \neq i$ is given by (3). Then, we can easily check that the resulting Markov chain is time-reversible with the following stationary distribution:

$$\pi(\underline{x}) \;\propto\; \exp(-\beta H(\underline{x})) \tag{4}$$

where

$$H(\underline{x}) = -\sum_{(i,j)\in E} x_i x_j - p \sum_{(i,j)\notin E} x_i x_j - (1-p)\sum_{i \in V} h_i x_i - p \sum_{i\in V} h_0 x_i.$$

Note that the stationary distribution of the state $+\underline{1}$ converges to 1 as $\beta \to \infty$. We can also verify that our game is a potential game with the potential function $H(\underline{x})$ [10].

Let $\underline{x}(t)$ be the state of the continuous Markov process at time $t \geq 0$. Denote by $T_+(\underline{w}) = \inf\{t \geq 0 : \underline{x}(t) = +\underline{1}, \underline{x}(0) = \underline{w}\}$, the hitting time on the state $+\underline{1}$, which measures the speed of diffusion to the equilibrium $+\underline{1}$. We also define the typical hitting time $\tau_+ = \tau_+(p,h)$ to be:

$$\tau_+ = \sup_{\underline{w}\in S} \inf \left\{ t \geq 0 \;\Big|\; \Pr(T_+(\underline{w}) > t) \leq \frac{1}{2} \right\}$$

For notational simplicity, we sometimes use just τ_+ unless confusion arises, but use $\tau_+(p,h)$ to emphasize the impact of p and h. Note that $\tau_+(0,h)$ is the typical hitting time without global information. Our interest is to study how $\tau_+(p,h)$ is affected by p, i.e., the impact of global information on the diffusion speed, which may depend on the topological structure of the underlying social network.

IV. Analysis

This section investigates how the typical hitting time on the state $+\underline{1}$ changes as p changes from 0 to 1. It is known [7] that the hitting time on the state $+\underline{1}$ is

$$\tau_+ = \tau_+(p,h) = \exp\left\{ \beta \tilde{\Gamma}_+(p,h) + o(\beta) \right\}, \tag{5}$$

where

$$\tilde{\Gamma}_+(p,h) = \max_{\underline{z}\neq +\underline{1}} \min_{\omega: \underline{z} \to +\underline{1}} \max_{t \leq |\omega|-1} H_{p,h}(\omega_t) - H_{p,h}(\underline{z}). \tag{6}$$

In the above, the minimum is taken over path $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$, i.e., ω_i and ω_{i+1} differ in a coordinate, starting from $\omega_1 = \underline{z}$ and ending with $\omega_m = +\underline{1}$ and $H_{p,h}(\cdot)$ means $H(\cdot)$ where we just emphasize its dependency on p, h. We first state a monotone property of $\tilde{\Gamma}_+$ as follows.

Theorem 1: $\tilde{\Gamma}_+(p,h)$ is a decreasing function with respect to h.

Proof: From the monotonicity of the Ising model [7], it follows that

$$\begin{aligned}
\tilde{\Gamma}_+(p,h) \quad &= \quad \max_{\underline{z}\neq +\underline{1}} \min_{\omega: \underline{z} \to +\underline{1}} \max_{t \leq |\omega|-1} H_{p,h}(\omega_t) - H_{p,h}(\underline{z}) \\
&= \quad \min_{\omega: -\underline{1} \to +\underline{1}} \max_{t \leq |\omega|-1} H_{p,h}(\omega_t) - H_{p,h}(-\underline{1}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&H_{p,h}(\underline{x}) - H_{p,h}(-\underline{1}) \\
&\quad = \quad (1-p)\sum_{(i,j)\in E}(1 - x_i x_j) + p \sum_{i,j: i\neq j}(1 - x_i x_j) \\
&\qquad\quad - \sum_{i\in V} h_i(x_i + 1) - p \sum_{i \in V}(h_0 - h_i)(x_i + 1)
\end{aligned}$$

Since $h_0 - h_i = h(N-1-N_i)$ and $x_i + 1 \geq 0$, $\tilde{\Gamma}_+(p,h)$ is decreasing with respect to h. ■

Theorem 1 implies that the big difference between the quality of the entrant and that of the incumbent results in fast diffusion of innovation, as we expect. In the following sections, we obtain bounds on p where the hitting time τ_+ (or equivalently $\tilde{\Gamma}_+$) undergoes a transition from '(constantly) small' to '(exponentially) large'.

A. Subcritical Regime: Small p

In this section, our goal is to obtain an upper bound of p so that the hitting time τ_+ remains comparable to that without global information (i.e., $p = 0$). To this end, we prove the following theorem.

Theorem 2: If $p \leq \min_{i \in V} \frac{h_i}{N}$, then for some $0 < h' < h$,

$$\tau_+(p,h) \leq \exp\{\beta \tilde{\Gamma}_+(0,h') + o(\beta)\}.$$

Proof: From $p \leq \min_{i\in V} \frac{h_i}{N}$, there exists $h' > 0$ such that

$$h_i - pN \geq h_i', \tag{7}$$

where $h_i' = h' N_i$.

For given $\underline{x}$, recall that N^+ be the number of +1 in $\underline{x}$ and note that $N^+ = \frac{1}{2}\sum_{i\in V}(x_i + 1)$. Then, we have

$$\begin{aligned} & p \sum_{i,j:i\neq j} (1 - x_i x_j) \\ &= p \cdot \frac{N(N-1)}{2} - p \sum_{i,j:i\neq j} x_i x_j \\ &= p \cdot \frac{N(N-1)}{2} - \frac{1}{2} p \left(\left(\sum_i x_i \right)^2 - \sum_i x_i^2 \right) \\ &= p \cdot \frac{N(N-1)}{2} - \frac{1}{2} p \left((N^+ - (N - N^+))^2 - N \right) \\ &= 2pN^+(N - N^+) \\ &\leq 2pN^+ N. \end{aligned} \tag{8}$$

Using the above inequality, it follows that

$$\begin{aligned} & H_{p,h}(\underline{x}) - H_{p,h}(-\underline{1}) \\ &= (1-p) \sum_{(i,j)\in E} (1 - x_i x_j) + p \sum_{i,j:i\neq j} (1 - x_i x_j) \\ &\quad - \sum_{i\in V} h_i (x_i + 1) - p \sum_{i\in V} (h_0 - h_i)(x_i + 1) \\ &\leq \sum_{(i,j)\in E} (1 - x_i x_j) + 2pN^+ N - \sum_{i\in V} h_i (x_i + 1) \\ &= \sum_{(i,j)\in E} (1 - x_i x_j) + \sum_{i\in V} pN(x_i + 1) - \sum_{i\in V} h_i (x_i + 1) \\ &= \sum_{(i,j)\in E} (1 - x_i x_j) - \sum_{i\in V} (h_i - pN)(x_i + 1) \\ &\leq \sum_{(i,j)\in E} (1 - x_i x_j) - \sum_{i\in V} h_i'(x_i + 1) \qquad (\because (7)) \\ &= H_{0,h'}(\underline{x}) - H_{0,h'}(-\underline{1}). \end{aligned}$$

Combining the above inequality with (5) and (6) leads to the conclusion of Theorem 2. ■

Note that h denotes the quality difference between the entrant and the imcumbent and that the convergence rate to the equilibrium $+\underline{1}$ gets slow with h by Theorem 1.

In [4], Ellison studied the asymptotic behavior of the convergence time to reach the state $+\underline{1}$ for a complete graph and a $2k$-regular graph where there is a link between node i and j if and only if $i - j \equiv \pm k \pmod N$ as $N \to \infty$. Ellison showed that $\tau_+ = O(1)$ for the $2k$-regular graph and $\tau_+ = O(\exp(N))$ for the complete graph. Ellison's results are extended in [7] where typical hitting times to the equilibirum $+\underline{1}$ are analyzed for arbitrary underlying graph structres. An interesting result of [7] is the case of a d-dimensional graph with bounded range [2] where for any h, the asymptotic behavior of typical hitting time τ_+ to $+\underline{1}$ is $O(1)$ as $N \to \infty$ (see Section 3 in [7]). Applying the above Theorem to d-dimensional graphs with bounded range we have the following corollary.

Corollary 3: For any d-dimensional graph G with bounded range,

$$\tilde{\Gamma}_+(p,h) \;=\; O(1), \qquad \text{if } \; p \leq \min_{i\in V} \frac{h_i}{2N}.$$

B. Supercritical Regime: Large p

Based on the results of [4] and [7], a well-connected graph structure hinders the fast diffusion of innovation. Inspired by this, in this subsection, we study the regime when p is large; we finds the asymptotic value of p above which the typical hitting time τ_+ becomes extremely large. The theorem below is the main result of this subsection and its implications will be described later.

Theorem 4: For any graph G with average degree Δ,

$$\tau_+(p,h) \;=\; \frac{\exp\left[\beta(N+1)\left(p\left(\frac{1}{2} - h\right)(N-1) - 2h\Delta\right)\right]}{O\left(N 2^N\right)}.$$

Proof: First, we consider a discrete-time version of the logit response dynamics. Instead of considering Poisson clocks, we revise the dynamics so that at discrete-time $t \in \mathbb{Z}_+$, a player is chosen uniformly at random and updates her strategy following the logit form (3). Let $\underline{z}(t)$ be the strategy profile of players at time $t \in \mathbb{Z}_+$ under this discrete-time dynamics. Then, $\{\underline{z}(t) : t \in \mathbb{Z}_+\}$ is a discrete-time reversible Markov chain on $\mathcal{S}$ with the same stationary distribution π as (4). Furthermore, one can define $T_+^{\text{disc}}(\underline{w}), \tau_+^{\text{disc}}$ for $\underline{z}(t)$ analogously as $T_+(\underline{w}), \tau_+$ for $\underline{x}(t)$. Furthermore, we have

$$T_+(\underline{w}) = \inf_t \left\{ \text{Poisson}(t) = T_+^{\text{disc}}(\underline{w}) \right\},$$

where $\{\text{Poisson}(t), t \geq 0\}$ is the Poisson process with rate N. This implies that

$$\tau_+^{\text{disc}} \;=\; O(N\tau_+) \tag{9}$$

Thus, from (9), it suffices to prove that

$$\tau_+^{\text{disc}} \;=\; \frac{\exp\left[\beta(N+1)\left(p\left(\frac{1}{2} - h\right)(N-1) - 2h\Delta\right)\right]}{O(2^N)}.$$

[2]In a d-dimensional graph with range K, any vertex i is assigned to a point $\xi_i \in \mathbb{R}^d$ such that if $(i,j) \in E$, then the Euclidean distance between ξ_i and ξ_j is less than K and any cube of volume v contains at most $2v$ vertices.

Furthermore, from the definition of τ_+^{disc}, it is enough to show that

$$\Pr\left(T_+^{\text{disc}} \le t \,\middle|\, \underline{z}(0) = -\underline{1}\right) \;<\; 1/2, \qquad (10)$$

for all

$$t < \frac{\exp\left[\beta(N+1)\left(p\left(\frac{1}{2}-h\right)(N-1) - 2h\Delta\right)\right]}{2^{N+1}}.$$

In the rest of the proof, we will implicitly assume $\underline{z}(0) = -\underline{1}$.

Now let P be the transition matrix for the discrete-time Markov chain. Define the set A as

$$A \;=\; \left\{\underline{z} \in \mathcal{S} : \sum_{i\in V} z_i \in \{0,1\}\right\}.$$

Then, we have that for $\underline{z} \in A$,

$$\frac{\pi(\underline{z})}{\pi(-\underline{1})} = \exp\left(\beta\left[-H(\underline{z}) + H(-\underline{1})\right]\right),$$

and

$$\begin{aligned}
&-H(\underline{z}) + H(-\underline{1})\\
&= \sum_{(i,j)\in E} (z_i z_j - 1) + p \sum_{(i,j)\notin E} (z_i z_j - 1)\\
&\qquad + 2(1-p) \sum_{i\in V: z_i = 1} h_i + 2p \sum_{i \in V: z_i = 1} h_0\\
&= (1-p) \sum_{(i,j)\in E} (z_i z_j - 1) + p \sum_{i\neq j} (z_i z_j - 1)\\
&\qquad + 2(1-p)h \sum_{i\in V: z_i = 1} N_i + 2p \sum_{i\in V: z_i=1} h(N-1)\\
&\le p\sum_{i\neq j}(z_i z_j - 1) + 2(1-p)h\cdot 2|E| + 2p\cdot\frac{N+1}{2}\cdot h(N-1)\\
&\le p\sum_{i\neq j}(z_i z_j - 1) + 2h(1-p)\Delta N + ph(N^2-1)
\end{aligned}$$

where we use $2|E| = \Delta N$ for the last inequality. Therefore, we have

$$\begin{aligned}
&-H(\underline{z}) + H(-\underline{1})\\
&\le p\sum_{i\neq j}(z_i z_j - 1) + 2h\Delta(N+1) + ph(N^2-1)\\
&\le -p\frac{N^2-1}{2} + 2h\Delta(N+1) + ph(N^2-1)\\
&= \frac{1}{2}(N+1)\left(p(N-1)(2h-1) + 4h\Delta\right)
\end{aligned}$$

where the second inequality follows from

$$2\sum_{i\neq j} z_i z_j \;=\; \left(\sum_i z_i\right)^2 - \sum_i z_i^2 \;\le\; 1 - N.$$

Let

$$g(N,h,\Delta) = \frac{1}{2}(N+1)\left(p(N-1)(2h-1) + 4h\Delta\right).$$

Now we observe that for any $t \in \mathbb{Z}_+$,

$$\begin{aligned}
\Pr\left(T_+^{\text{disc}} \le t\right) &= \Pr\left(\underline{z}(s) = +\underline{1}, \quad \text{for some } s \le t\right)\\
&\le \Pr\left(\underline{z}(s) \in A, \quad \text{for some } s\le t\right)\\
&\le \sum_{s=1}^{t}\sum_{\underline{z}\in A} \Pr\left(\underline{z}(s) = \underline{z}\right)\\
&= \sum_{s=1}^{t}\sum_{\underline{z}\in A} \left(P^t\right)_{-\underline{1},\underline{z}},
\end{aligned}$$

where the second inequality is from the union bound and the last equality holds by the assumption $\underline{z}(0) = -\underline{1}$. Then,

$$\begin{aligned}
&\Pr\left(T_+^{\text{disc}} \le t\right)\\
&= \sum_{s=1}^{t}\sum_{\underline{z}\in A}\left(P^t\right)_{-\underline{1},\underline{z}}\\
&= \sum_{s=1}^{t}\sum_{\underline{z}\in A}\frac{\pi(\underline{z})}{\pi(-\underline{1})}\left(P^t\right)_{\underline{z},-\underline{1}} \quad (\because \text{revesability of } P)\\
&\le \sum_{s=1}^{t}\sum_{\underline{z}\in A}\frac{\pi(\underline{z})}{\pi(-\underline{1})} \;\le\; \sum_{s=1}^{t}\sum_{\underline{z}\in A} e^{\beta g(N,h,\Delta)}\\
&\le t2^N e^{\beta g(N,h,\Delta)}. \qquad (\because |A| \le 2^N)
\end{aligned}$$

Therefore, the probability is less than $1/2$ if

$$t \;<\; 2^{-N-1}e^{-\beta g(N,h,\Delta)}.$$

This completes the proof of (10) and that of Theorem 4. ■

The above theorem implies that for large enough N, τ_+ is exponentially large with respect to N (i.e., the diffusion is quite slow) *regardless of* the underlying graph structure if

$$p\,(1-2h)\,(N-1) - 4h\Delta - \frac{2\log 2}{\beta} > 0. \qquad (11)$$

Equivalently, we have the following corollary.

Corollary 5: For any graph G with average degree Δ,

$$\tau_+ = \exp(\Omega(\varepsilon N)),$$

if $h < \frac{1}{2}$ and $p > \frac{4h\Delta + \frac{2\log 2}{\beta} + \varepsilon}{(1-2h)(N-1)}$ for some $\varepsilon > 0$.

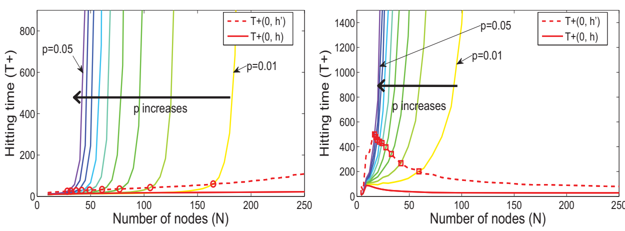
Corollary 3 and Corollary 5 have the following implication for the d-dimensional graphs with $h < 1/2$: there exist constants c_1, c_2 (depending on d, h) such that

$$\text{the typical hitting time } \tau_+ \text{ is } \begin{cases}\exp(\Omega(\varepsilon N)) & \text{if } p > c_1/N.\\ \exp(O(\beta)) & \text{if } p < c_2/N\end{cases}.$$

V. Numerical Results

In this section, we show numerical results with focus on the hitting time on the state of $+\underline{1}$ for d-dimensional graphs and a real on-line social network, where we consider two typical classes of d-dimensional graphs: rings and lattices [7]. For the real online social network, we use "ego-Facebook" traces from Stanford large network dataset collections [11]. In all plots, we show the average of 1000 simulations, and all plots start from the initial state $-\underline{1}$.

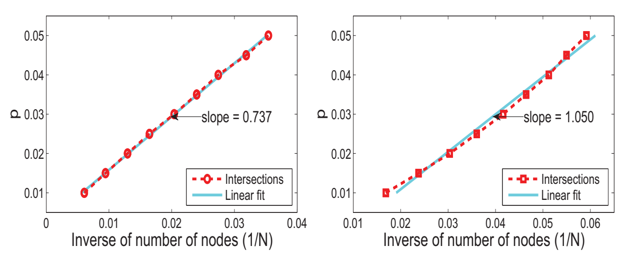
Fig. 1 shows how the hitting times changes for rings and lattices when p varies from 0.01 to 0.05. Rings are chosen to verify the results of Ellison [4]. We use $a = 10, b = 5, c = d = 0$ (which results in $h = 1/3$). The network size in rings ranges from 5 to 250, and we use two lattices: l by l and l by $l + 1$, where l varies from 2 to 16 so that the size of resulting lattices varies from 4 to 256. We observe that in both rings and lattices, the hitting times increase with p and is asymptotically bounded by that for $p = 0$, as $N \to \infty$, as theoretically discussed in Section IV.



(a) Ring (b) Lattice

Fig. 1. Hitting time over rings and lattices (p varies from 0.01 to 0.05.)

In Section IV, we discussed that there is a phase transition threshold for the asymptotic value of p, where above the order of $1/N$, the diffusion speed is slow and below the order of $1/N$, the diffusion speed is bounded by that with no global information and smaller h' than h. To investigate such phase transition effect, in Fig. 2 we plot the points where $T_+(p, h)$ and $T_+(0, h/2)$ intersect. Note that in Fig. 1, above $T_+(0, h/2)$, the diffusion speed is slow and below $T_+(0, h/2)$, the diffusion speed is close to $T_+(0, h)$. Fig. 2 shows that the intersection points are approximately proportional to $1/N$, which verifies our result that the phase transition in the hitting time occurs when p is in the order of $1/N$, as stated in Theorems 2 and 4.



(a) Ring (b) Lattice

Fig. 2. Intersections of $T_+(0, h')$ and $T_+(p, h)$ (p varies from 0.01 to 0.05).

Fig. 3 exhibits the hitting times on $+\underline{1}$ for the ego-Facebook network. We found that the ego-Facebook network is not connected. Thus, we extracted the connected giant components and used them in our numerical results. The statistical properties of the graphs are listed in Table II. Here, we use $a = 15, b = 3, c = d = 0$ (thus, $h = 2/3$). As in the rings and lattices, we observe that the hitting time increases with p. In general, as the network size grows, the hitting time naturally increases.

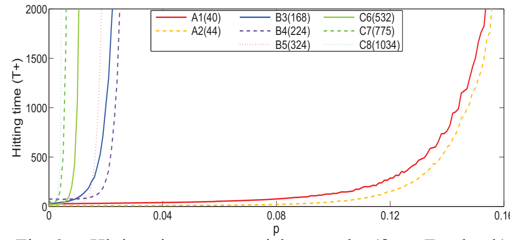


Fig. 3. Hitting time over social networks (from Facebook)

TABLE II
STATISTICAL PROPERTIES OF THE SOCIAL NETWORKS

ID	number of nodes	total number of edges	mean degree
A1	40	220	11.00
A2	44	138	6.27
B3	168	1656	19.71
B4	224	3192	29.50
B5	324	2514	15.52
C6	532	4812	18.04
C7	775	14024	35.68
C8	1034	26749	51.74

VI. Conclusion

This paper studied how global information given as a collective signal slows down the spread of new technology when users make rational decision in the presence of noise. We have found that global information impedes the diffusion of innovations whenever the (asymptotic) amount of global information exceeds some critical threshold. As an interesting future work, we plan to analyze the convergence speed in an epidemic model and compare the impacts of global information over two diffusion models.

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