Optimality Certificate of Dynamic Spectrum Management in Multi-Carrier Interference Channels

Paschalis Tsiaflakis[1], Chee Wei Tan[2], Yung Yi[2], Mung Chiang[2], Marc Moonen[1]
[1] Electrical Engineering, Katholieke Universiteit Leuven, Belgium
[2] Electrical Engineering, Princeton University, USA

***Abstract*— The multi-carrier interference channel where interference is treated as additive white Gaussian noise, is a very active topic of research, particularly important in the area of Dynamic Spectrum Management (DSM) for Digital Subscriber Lines (DSL). Here, multiple users optimize their transmit power spectra so as to maximize the total weighted sum of data rates. The corresponding optimization problem is however nonconvex and thus computationally intractable, i.e. a certificate of global optimality requires exponential time complexity algorithms. This paper shows that under certain channel conditions, this nonconvex problem can be solved in polynomial time with a certificate of global optimality. The channel conditions are discussed consisting of different interference models including synchronous and asynchronous DSL transmission. Simulations demonstrate its applicability to realistic DSL scenarios.**

I. Introduction

Digital Subscriber Line (DSL) technology remains the number one choice of broadband access technology in the world [1]. The corresponding increased demand for greater bandwidth and faster connection has led to several technological approaches to mitigate the channel impairments. This has become one of the most important applications of the multi-carrier interference channel model in today's Internet.

One of the major limitations of DSL performance today is crosstalk among different lines operating in the same cable bundle. Dynamic Spectrum Management (DSM) refers to a set of solutions to the crosstalk problem. Basically these solutions consist of signal level coordination and/or spectrum level coordination. For example, receiver and transmitter signal level coordination correspond to a multiple access channel and broadcast channel respectively. In this paper the focus is on spectrum level coordination, which is also referred to as spectrum management, spectrum balancing, or multi-carrier power control.

Spectrum level coordination corresponds to a multi-carrier interference channel where a number of different users try to communicate their separate information over a coupled (wired) channel. The capacity region of the interference channel is still an open problem, but for practical systems where the crosstalk channels are typically weaker than the direct channel, treating interference as additive white Gaussian noise has been the most practical communication strategy in operational networks. In this case the transmit power spectrum of each user is designed so as to minimize interference to other lines, while maintaining a satisfactory data rate. Such a technique reduces the impact of crosstalk and significantly improves the overall data rate beyond those of the current approach of static spectrum management where fixed over-conservative transmit power spectra are used.

The problem of optimally choosing the transmit power spectra in order to maximize the total weighted sum of data rates of the users turns out to be a *nonconvex optimization problem*. Several solutions have been proposed ranging from centralized to distributed algorithms, e.g., [2] [3] [4] [5] [6]. However, there is an undesirable gap in the analysis of these DSM algorithms: we either have algorithms with optimality certificate but exponential running-time, or polynomial-time solutions with only numerical verification of small suboptimality gap but no theoretical characterization of optimality condition.

Indeed, on the one hand, polynomial-time algorithms like IW [5], ISB [3] and ASB [4] have only sufficient conditions for convergence but not for optimality. On the other hand, Optimal Spectrum Balancing (OSB) [2], Branch and Bound Optimal Spectrum Balancing (BB-OSB) [7] and a prismatic branch and bound algorithm (PBB) [8], provide a certificate of finding the globally optimal solution, up to a certain accuracy depending on the used discretization of the search space. Unfortunately, the complexity of these algorithms grows exponentially with the number of users. Furthermore most of the aformentioned algorithms assume that the number of frequency tones is infinitely large so that the Lagrangian duality gap can be viewed as zero [9].

In this paper, we give the conditions on the channel, noise, and power constraints, under which this difficult nonconvex problem of DSM can be solved in polynomial time with an optimality certificate. To the best of our knowledge, our results provide the first sufficient condition for optimality for a finite number of carriers, and it is applicable for a general model including synchronous as well as asynchronous [10] DSL transmission. There are two key innovations in our proofs: the use of geometric programming (GP) and M-matrix theory. Furthermore, the optimality analysis further leads to a new DSM algorithm, with simulations demonstrating the applicability of this optimality certificate to realistic DSL scenarios.

The following notations are used. Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors and italics denote scalars. Furthermore $\mathbf{x} \succeq \mathbf{y}$ denotes componentwise inequality between vectors $\mathbf{x}$ and $\mathbf{y}$ and $\mathbf{X} \succeq \mathbf{Y}$ denotes componentwise inequality between matrices $\mathbf{X}$ and $\mathbf{Y}$. We also let $(\mathbf{x})_l$ denote the lth element of $\mathbf{x}$ and $[\mathbf{X}]_{ij}$ denote the element of matrix X on row i and column j. We denote the identity matrix by $\mathbf{I}$ and the spectral radius of $\mathbf{T}$

by $\rho(\mathbf{T})$.

II. System Model

Most current DSL systems use Discrete Multi-Tone (DMT) modulation. The basic idea of DMT is to split the available bandwidth into a large number of frequency bands, also called tones. For each tone, a transmit power can be allocated individually which corresponds to a number of transmitted bits. The total data rate for each user is then obtained by adding its transmitted bits over all tones.

In our model, no signal coordination is assumed between the transmitting and/or receiving modem for each user. Each user regards the signals from the other users as noise. The only degree of freedom is choosing the transmit powers of the different users over the different tones.

The transmit power is denoted as s_k^n, where n is the user index ranging from 1 to N and k is the tone index ranging from 1 to K. The noise power is denoted as σ_k^n. The vector containing the transmit powers of user n on all tones is $\mathbf{s}^n \triangleq [s_1^n, s_2^n, \ldots, s_K^n]^T$. $h_k^{n,m}$ denotes the squared channel gain magnitude from user m into user n on tone k, where it refers to the squared channel gain magnitude of line n when $m = n$. Furthermore int_k^n is the interference caused to user n on frequency tone k. The achievable bit loading for user n on tone k can then be expressed as

$$b_k^n \triangleq \log_2\left(1 + \frac{h_k^{n,n} s_k^n}{\Gamma(\text{int}_k^n + \sigma_k^n)}\right) \text{ bits/ s/ Hz}, \quad (1)$$

where Γ denotes the signal-to-noise ratio (SNR) gap to capacity, which is a function of the desired bit error ratio (BER), the coding gain and noise margin [11].

The DMT symbol rate is denoted as f_s. The total data rate for user n and the total power used by user n are then, respectively, given by

$$R^n = f_s \sum_k b_k^n \qquad \text{and} \qquad P^n = \sum_k s_k^n. \quad (2)$$

Such a model can be used to analyze the following special cases.

A first case is synchronized DSL transmission. Here it is assumed that all users are aligned in frequency so that each tone is capable of transmitting data independently from the other tones. The interference caused to user n on tone k can then be expressed as $\text{int}_k^n = \sum_{m \neq n} h_k^{n,m} s_k^m$.

A second case is the asynchronous DSL transmission case [10]. Here there is not only crosstalk within one tone but also between different tones of different users. The interference caused to user n on tone k can then be expressed as $\text{int}_k^n = \sum_{m \neq n}(\sum_{p=1}^K h_{k,p}^{n,m} s_p^m)$ where $h_{k,p}^{n,m}$ refers to the squared channel gain magnitude from user m tone p into user n tone k.

A third case is a general case that includes all the previous cases and equals $\text{int}_k^n = \sum_{m=1}^N(\sum_{p=1}^K h_{k,p}^{n,m} s_p^m) - h_{k,k}^{n,n}$, where $h_{k,p}^{n,m}$ is the squared channel gain magnitude from user m tone p into user n tone k.

All these cases can be summarized with the following bit loading expression for user n on tone k

$$b_k^n = \log_2\left(1 + \frac{h_{k,k}^{n,n} s_k^n}{\sum_{m=1}^N \sum_{p=1}^K h_{k,p}^{n,m} s_p^m - h_{k,k}^{n,n} s_k^n + \Gamma\sigma_k^n}\right). \quad (3)$$

For each of the above cases the parameters $h_{k,p}^{n,m}$ have their specific values. In the rest of this paper we will focus on this general case (3) so that the results can be used for all three cases. Note that for notational simplicity, we absorb Γ into the definition of $h_{k,p}^{n,m}$.

III. Spectrum Management Problems

The key goal in this text is to maximize the data rates of the bundle of interfering DSL lines. To this end, the objective is to find the optimal transmit power spectra maximizing a weighted sum of data rates subject to power constraints. Assuming a per-user total power constraint P^n for each user n, this is formulated as the following problem:

$$\begin{array}{ll} \max_{\mathbf{s}^1,\ldots,\mathbf{s}^N} \sum_{n=1}^N w_n R^n & \\ \text{s.t.} \sum_{k=1}^K s_k^n \leq P^n & , \quad \forall n, \\ \text{s.t. } 0 \leq s_k^n, \quad \forall n, \forall k. & \end{array} \quad (4)$$

Instead of per-user total power constraints, we will consider one total power constraint P on all users, where $P = \sum_{n=1}^N P^n$. Using (2) this can be formulated as follows:

$$\begin{array}{ll} \max_{\mathbf{s}^1,\ldots,\mathbf{s}^N} \sum_{n=1}^N w_n f_s \sum_{k=1}^K b_k^n & \\ \text{s.t.} \sum_{k=1}^K \sum_{n=1}^N s_k^n \leq P & , \\ \text{s.t. } 0 \leq s_k^n, \quad \forall n, \forall k, & \end{array} \quad (5)$$

where b_k^n is given by the general interference model (3). Note that (4) and (5) are both nonconvex, and thus intractable. Also note that in (5) the inequality total power constraint can be replaced by an equality. This is justified by lemma 1.

Lemma 1: At optimality, we have $\sum_{k=1}^K \sum_{n=1}^N s_k^n = P$ in (5).

Proof: Suppose the optimal solution to (5) results in a total power P^{t} where $P^{\text{t}} < P$. As this is the optimal solution it means that no better solution can exist. This is not true because by scaling all the powers s_k^n by a factor P/P^{t}, the objective function in (5) increases. Thus, at optimality, $P^{\text{t}} = P$ in (5). ■

Note that Problem (5) can be seen as a relaxation of problem (4). For symmetric crosstalk scenarios with not too large crosstalk (cf. section IV) the optimal solution of (5) allocates equal powers to all users and so solving (5) and (4) leads to exactly the same solution, i.e. zero relaxation gap. For asymmetric crosstalk scenarios, the users will not necessarily be allocated the same amount of total transmit powers and there may be a relaxation gap between the solution of (4) and (5). The solution of (5) can however provide a useful upper bound for problem (4).

Finally, we note that the nonconvex spectrum management problem with one total power constraint on all users (5) has also been considered in other work [12].

IV. Optimality certificate for polynomial time spectrum management

A. Polynomial time spectrum management

Since (5) is nonconvex, it can have many locally optimal solutions depending on the values of the channel and noise coefficients. In order to solve this nonconvex problem with a certificate of global optimality, i.e., the solution is globally optimal, one has to resort to exponential time complexity algorithms such as [12] [2]. In Theorem 1, we provide sufficient conditions on the channel and noise coefficients, and the total power constraint for which (5) can be solved in polynomial time with a certificate of global optimality. In prior to describing Theorem 1, we first define the matrix $\mathbf{B}$ as follows:

$$[\mathbf{B}]_{d,l} = (\frac{h_{k,p}^{n,m}}{h_{k,k}^{n,n}} + \frac{\Gamma\sigma_k^n}{P h_{k,k}^{n,n}}). \tag{6}$$

where

$$l = n + (k-1)N, \quad d = m + (p-1)N.$$

Theorem 1: For the general interference system model (3), (5) can be solved optimally in polynomial time when

$$\mathbf{B} \text{ is nonsingular and } \mathbf{C} \triangleq \mathbf{I} - \mathbf{B}^{-1} \succeq \mathbf{0}. \tag{7}$$

Proof: We start from b_k^n of (5) by incorporating the total power equality constraint $\sum_{k=1}^{K}\sum_{n=1}^{N} s_k^n = P$ (see Lemma 1) as follows:

$$b_k^n = \log_2\left(1 + \frac{h_{k,k}^{n,n} s_k^n}{\sum_{m=1}^{N}\sum_{p=1}^{K} h_{k,p}^{n,m} s_p^m - h_{k,k}^{n,n} s_k^n + \Gamma\sigma_k^n \frac{\sum_{m=1}^{N}\sum_{p=1}^{K} s_p^m}{P}}\right) \tag{8}$$

This is reorganized as follows:

$$b_k^n = \log_2\left(\frac{\sum_{m=1}^{N}\sum_{p=1}^{K}(\frac{h_{k,p}^{n,m}}{h_{k,k}^{n,n}} + \frac{\Gamma\sigma_k^n}{h_{k,k}^{n,n}P}) s_p^m}{\sum_{m=1}^{N}\sum_{p=1}^{K}(\frac{h_{k,p}^{n,m}}{h_{k,k}^{n,n}} + \frac{\Gamma\sigma_k^n}{h_{k,k}^{n,n}P}) s_p^m - s_k^n}\right). \tag{9}$$

Using the symbol $\tilde{B}_{k,p}^{n,m} = (\frac{h_{k,p}^{n,m}}{h_{k,k}^{n,n}} + \frac{\Gamma\sigma_k^n}{P h_{k,k}^{n,n}})$, this can be simplified by

$$b_k^n = \log_2\left(\frac{\sum_{m=1}^{N}\sum_{p=1}^{K}\tilde{B}_{k,p}^{n,m} s_p^m}{\sum_{m=1}^{N}\sum_{p=1}^{K}\tilde{B}_{k,p}^{n,m} s_p^m - s_k^n}\right). \tag{10}$$

As it is not so easy to represent four-dimensional matrices, we go to a two-dimensional matrix by a bijection map $\{l,k\} \leftrightarrow \{n,m,k,p\}$ as follows:

$$\begin{aligned} l &= n + (k-1)N \\ d &= m + (p-1)N \\ [\mathbf{B}]_{d,l} &= \tilde{B}_{k,p}^{n,m} \end{aligned} \tag{11}$$

$$\sum_{n=1}^{N}\sum_{k=1}^{K}\tilde{B}_{k,p}^{n,m} s_k^n = \sum_{l=1}^{L=KN}[\mathbf{B}]_{d,l} s_l, \tag{12}$$

Now, (5) can be reformulated as follows:

$$\begin{array}{ll} \max_{s_1 \dots s_L} & \sum_{l=1}^{L=KN} w_l \log_2\left(\frac{\sum_{q=1}^{L}[\mathbf{B}]_{l,q}s_q}{\sum_{q=1}^{L}[\mathbf{B}]_{l,q}s_q - s_l}\right) \\ \text{s.t.} & \sum_{l=1}^{L} s_l = P \\ \text{s.t.} & s_l \geq 0 \quad \forall l \end{array} \tag{13}$$

where $w_l = w_{\text{rem}(l-1,N)+1}$ and rem(a,b) refers to the remainder after dividing a by b.

By using condition (7) we can make the following change of variables:

$$\mathbf{y} = \mathbf{B}\mathbf{s}, \qquad \mathbf{s} = \mathbf{B}^{-1}\mathbf{y} = \mathbf{y} - \mathbf{C}\mathbf{y}. \tag{14}$$

If (14) is substituted in (13) the cost function reduces to

$$\sum_{l=1}^{L} w_l \log(y_l/(\mathbf{C}\mathbf{y})_l) = \log(\prod_{l}^{L}(y_l^{w_l}/(\mathbf{C}\mathbf{y})_l^{w_l})), \tag{15}$$

and the last constraint to

$$y_l \geq (\mathbf{C}\mathbf{y})_l, \qquad l = 1, \dots, n. \tag{16}$$

We thus obtain the following optimization problem:

$$\begin{array}{ll} \min_{\mathbf{y}} & \prod_l((\mathbf{C}\mathbf{y})_l y_l^{-1})^{w_l} \\ \text{s.t.} & (\mathbf{C}\mathbf{y})_l y_l^{-1} \leq 1, \qquad l = 1, \dots, n \\ & \sum_l(\mathbf{B}^{-1}\mathbf{y})_l = P. \end{array} \tag{17}$$

Note that (15) and (16) are homogeneous in $\mathbf{y}$ (and so in $\mathbf{s}$). Thus the constraint $\sum_l(\mathbf{B}^{-1}\mathbf{y})_l = P$ or, equivalently, $\sum_l^L s_l = P$ acts as a normalization instead of a constraint. As a consequence it can be replaced by another normalization on $\mathbf{y}$ without changing the optimal value of (17), e.g.,

$$\prod_l y_l = 1. \tag{18}$$

This leads us to the following equivalent optimization problem:

$$\begin{array}{ll} \min_{\mathbf{y}} & \prod_l((\mathbf{C}\mathbf{y})_l y_l^{-1})^{w_l} \\ \text{s.t.} & (\mathbf{C}\mathbf{y})_l y_l^{-1} \leq 1, \qquad l = 1, \dots, L \\ & \prod_l y_l = 1. \end{array} \tag{19}$$

This can be reformulated as the following standard GP [14]:

$$\begin{array}{ll} \min_{\mathbf{y},\mathbf{t}} & \prod_l(t_l)^{w_l} \\ \text{s.t.} & (\mathbf{C}\mathbf{y})_l y_l^{-1} t_l^{-1} \leq 1, \qquad l = 1, \dots, L \\ & (\mathbf{C}\mathbf{y})_l y_l^{-1} \leq 1, \qquad l = 1, \dots, L \\ & \prod_l y_l = 1. \end{array} \tag{20}$$

For a particular obtained $\mathbf{y}$ for (20), we can recover the optimizer $\mathbf{s}$ of (5) by first performing

$$\mathbf{s} = (\mathbf{I} - \mathbf{C})\mathbf{y} \tag{21}$$

and then normalize (scale) this $\mathbf{s}$ such that it satisfies $\mathbf{1}^T\mathbf{s} = P$.

So when condition (7) is satisfied, the solution of the nonconvex spectrum optimization problem (5) can be found by solving a GP given by (20). A GP can be turned into a convex optimization problem so that a local optimum is also a global optimum. Furthermore the global optimum can be computed very efficiently. Numerical efficiency holds both in theory and in practice: interior point methods applied to GP have provably polynomial time complexity [15]. This concludes the proof of Theorem 1. ∎

B. Optimality Analysis

In section IV-A, we provided condition (7) under which nonconvex problem (5) can be solved in polynomial time with a certificate of global optimality. In this section we will further study this optimality condition. To this end, we will first introduce some definitions from matrix theory.

Definition 1: Any matrix $\mathbf{Q}$ of the following form:

$$\mathbf{Q} = t\mathbf{I} - \mathbf{T}, \;\; t > \rho(\mathbf{T}), \;\; \mathbf{T} \succeq \mathbf{0}, \tag{22}$$

is called an M-matrix [13].

Definition 2: A nonsingular nonnegative matrix $\mathbf{Q}$ is said to be an inverse M-matrix if $\mathbf{Q}^{-1}$ is an M-matrix.

Using the above definitions, condition (7) can be reformulated as follows: $\mathbf{B}$ is an inverse M-matrix with $t = 1$ where

t is defined in Definition 1. Generally this means that the off-diagonal elements of $\mathbf{B}^{-1}$ have to be negative and the diagonal elements of $\mathbf{B}^{-1}$ have to be smaller than 1.

Characterizing all matrices whose inverses are M-matrices is an active research field in the mathematical society, but there are only limited results, even for symmetric matrices [16]. In [16] one special type of matrix is introduced, called *generalized ultrametric matrix*, which is an inverse M-matrix. However this matrix does not necessarily satisfy the extra condition that $t = 1$. Thus for an N-user case with K frequency tones, it is difficult provide an exact meaning of these conditions on the channel and noise coefficients.

However, for a 2-user case with a single frequency tone, we can develop a sufficient condition on the channel and noise coefficients, and the total power, in order for the condition (7) to hold, i.e., $\mathbf{B}$ is an inverse M-matrix with $t = 1$. It also gives us an intuitive understanding of condition (7) for the 2-user case.

Theorem 2: For a 2-user DSL scenario with a single frequency tone, if

$$\Gamma h_1^{12}\,\Gamma h_1^{21} + \Gamma h_1^{21}\,\frac{\Gamma\sigma_1^1}{P} + \Gamma h_1^{12}\,\frac{\Gamma\sigma_1^2}{P} \le \min\left(h_1^{11}\frac{\Gamma\sigma_1^2}{P}, h_1^{22}\frac{\Gamma\sigma_1^1}{P}\right) \tag{23}$$

then condition (7) is satisfied.

Proof: For a 2-user DSL scenario with a single frequency tone, $\mathbf{B}$ is a 2×2 matrix. Now, (7) can be expressed in terms of the elements of $\mathbf{B}$ as

$$\mathbf{I} - \mathbf{B}^{-1} \succeq 0 \quad \Leftrightarrow \quad \begin{array}{r} \frac{b_{12}}{b_{11}b_{22}-b_{12}b_{21}} \ge 0, \\ \frac{b_{21}}{b_{11}b_{22}-b_{12}b_{21}} \ge 0, \\ 1 - \frac{b_{11}}{b_{11}b_{22}-b_{12}b_{21}} \ge 0, \\ 1 - \frac{b_{22}}{b_{11}b_{22}-b_{12}b_{21}} \ge 0. \end{array} \tag{24}$$

Since the elements of $\mathbf{B}$ are all positive, (24) implies

$$\max(b_{11}, b_{22}) \le b_{11}b_{22} - b_{12}b_{21}, \tag{25}$$

which can be rewritten in terms of the channel coefficients, noise and total power P:

$$\Gamma h_1^{12}\,\Gamma h_1^{21} + \Gamma h_1^{21}\,\frac{\Gamma\sigma_1^1}{P} + \Gamma h_1^{12}\,\frac{\Gamma\sigma_1^2}{P} \le \min\left(h_1^{11}\frac{\Gamma\sigma_1^2}{P}, h_1^{22}\frac{\Gamma\sigma_1^1}{P}\right). \tag{26}$$

■

Intuitively, condition (7) is satisfied if both the crosstalk (i.e. h_1^{21}, h_1^{12}) and the total power constraint P are not exceedingly large. The same intuition can be expected to hold for the case where there are multiple users and frequency tones, where developing mathematical conditions is left as a future work.

C. Algorithm DSM-GP

Next, we turn from the optimality analysis based on GP and M-matrix theory to the design of our DSM algorithm, Algorithm DSM-GP. Given the channel, noise and power constraint parameters of the DSL scenario, (7) in Theorem 1 can be checked easily. If (7) in Theorem 1 is true, we can solve (5) with global optimality by first solving (20) and then transforming back to the transmit powers (21) after the renormalization step.

Suppose that (7) in Theorem 1 is violated, but there are only a small percentage of negative C_{ij} elements. In this case, we can set the negative elements to be zero, resulting in the matrix $\tilde{C}_{ij} = \max\{C_{ij}, 0\}$. Intuitively, solving (20) using $\tilde{\mathbf{C}}$ should still lead to a near-optimal performance when the percentage of negative elements is reasonably small. This intuition is shown to be very useful in the next section, although proving the continuity of optimized objective function in the perturbation of C_{ij} remains an open issue. The obtained solution can then projected back to the feasible domain of (5) by using proportional adjustment of the transmit power to satisfy Lemma 1. However, the objective in (5) thus obtained using $\tilde{\mathbf{C}}$ is no longer globally optimal, and does not upper bound the global optimal objective of (4). We propose the following algorithm to solve (5):

Algorithm 1 DSM-GP Algorithm

Step 1 Calculate $\mathbf{C} = \mathbf{I} - \mathbf{B}^{-1}$.
Step 2 Force negative elements of $\mathbf{C}$ to zero leading to $\tilde{\mathbf{C}}$.
Step 3 Solve (20) using a GP solver (e.g., [14]) with $\tilde{\mathbf{C}}$.
Step 4 Transform solution $\mathbf{y}$ to transmit powers $\mathbf{s}$ using $\mathbf{s} = (\mathbf{I} - \tilde{\mathbf{C}})\mathbf{y}$ where negative elements of $\mathbf{s}$ are forced to zero.
Step 5 Scale $\mathbf{s}$ proportionally such that the total power of all users is equal to P (cf. Lemma 1)

V. Simulation Results

In this section simulation results will be shown for Algorithm 1 proposed in section IV for synchronous as well as asynchronous symmetric N-user ADSL downstream scenarios.

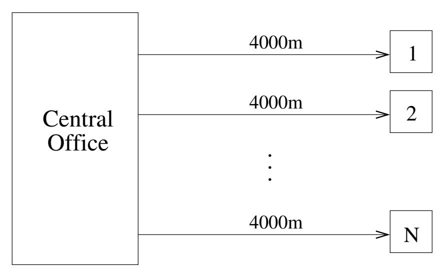

Fig. 1. Symmetric N-user ADSL downstream scenario

The ADSL downstream scenario is shown in Figure 1. The simulations are performed for a two-user case ($N = 2$) up to an eight-user case ($N = 8$). The four-user scenario, for example, consists of active users 1,2,3,4 where users 5,6,7,8 are inactive. The twisted pair lines have a diameter of 0.5 mm (24 AWG). The maximum transmit power is 20.4 dBm [17]. The SNR gap Γ is 12.9 dB, corresponding to a coding gain of 3 dB, a noise margin of 6 dB and a target symbol error probability of 10^{-7}. The tone spacing Δ_f is 4.3125 kHz. The DMT symbol rate f_s is 4 kHz. The simulations are performed in Matlab on a TravelMate 4002WLMi with 768 MB of RAM and an Intel Pentium M processor 1.60 GHz.

In Table I the simulation results are shown for the synchronous DSL transmission case, i.e. no intercarrier interference (ICI). The first column denotes the number of users. The second column denotes the performance of Algorithm 1 with respect to the global optimal algorithm OSB with a very fine granularity. The third and fourth column denote the number and percentage of negative C_{ij} elements for condition (7) respectively.

The simulation time of OSB requires exponential complexity, i.e. 3 minutes, 3 hours and 5 days for 2,3 and 4 users scenarios respectively. For more than four users it is infeasible to execute within acceptable time. Therefore the performance for more than 4 users can not be compared to OSB. The simulation time of Algorithm 1 just requires a few seconds even for the 8-user scenario.

Note that condition (7) is satisfied up to the 6-user scenario. This means that Algorithm 1 succeeds in finding the optimal transmit spectra with a certificate of global optimality for up to the 6-user scenario in a few seconds. Although the conditions are not satisfied for the 7- and 8-user scenarios, it has been verified that Algorithm 1 leads to near-optimal transmit spectra. That is because the percentage of negative elements is extremely small and forcing these to zero does not significantly change the problem. However there is no guarantee of global optimality. As the percentage of negative C_{ij} elements increases one can expect suboptimal transmit spectra.

In Table II the simulation results are shown for the asynchronous DSL transmission case [10], i.e. with intercarrier interference (ICI). Condition (7) is satisfied up to the 6-user scenario. For the 7- and 8-user scenarios there is a small percentage of negative C_{ij} elements. This percentage is larger than for the synchronous case because there is a larger amount of crosstalk caused by ICI. As there exists no globally optimal method for asynchronous DSL transmission, the performance of Algorithm 1 cannot be compared to such a benchmark. However for the 2- up to the 6-user case we still have a certificate of global optimality for Algorithm 1. This shows another powerful application of an optimality certificate in DSM algorithm analysis. In fact, since the channel conditions are symmetric, the optimal solution computed using Algorithm 1 for (5) is equal to (4).

TABLE I
ANALYSIS PERFORMANCE ALGORITHM 1: SYNCHRONOUS DSL TRANSMISSION

Users	Perf. wrt OSB	Nb. neg. $\mathbf{C}$ elem.	Perc. neg. $\mathbf{C}$ elem.
2	100.0%	0	0
3	100.0%	0	0
4	100.0%	0	0
5	/	0	0
6	/	0	0
7	/	140	0.0057%
8	/	352	0.011%

TABLE II
ANALYSIS PERFORMANCE ALGORITHM 1: ASYNCHRONOUS DSL TRANSMISSION

Users	Nb. neg. $\mathbf{C}$ elem.	Perc. neg. $\mathbf{C}$ elem.
2	0	0
3	0	0
4	0	0
5	0	0
6	0	0
7	203	0.0083%
8	392	0.012%

VI. CONCLUSION

Maximizing the weighted date rate of all users for a multi-carrier interference channel is an important problem that finds application in DSL. DSL spectrum management has attracted many researchers who produced a large variety of algorithms. However, since the problem is nonconvex, all the DSM algorithms that produce an optimality certificate require exponential-time complexity. Also, most previous suboptimal DSM heuristics give only sufficient conditions for convergence to a suboptimal solution. This paper leverages on GP and M-matrix theory to generate a polynomial-time optimality certificate for a set of conditions on channel, noise and power constraints. Our results shed new light on the structure of power allocation across multiple carriers in DSL interference channels. Furthermore, we propose Algorithm DSM-GP that can compute the global optimal transmit power allocation. This paper provides another step towards understanding the tradeoff between DSM algorithm's complexity and performance guarantee.

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